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EXACT SOLUTIONS AND MATHEMATICAL PROPERTIES OF BOUNDARY-VALUE PROBLEMS FOR DYNAMIC-DIFFUSION BOUNDARY LAYERS

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The paper studies boundary-value problems for dynamic-diffusion boundary layers occurring near a vertical wall at high Schmidt numbers and for dynamic boundary layers whose inner edge is adjacent to the dynamic-diffusion layers. Exact solutions for boundary layers at small and large times are derived. The well-posedness of the boundary-value problem for a steady dynamic-diffusion layer is studied.

We consider the free convection in a viscous fluid near a vertical wall (support) and impurity transport in the case of growth of a thin film from a solution–melt of semiconducting materials. In this case, the geometry of the region can be considered constant because the growth rate is assumed to be negligibly small (in practice, the film thickness increases by 10 μm for 1 h). If the average impurity concentration in the solution is not equal to the equilibrium concentration, a zone of reduced impurity concentration is formed near the support (in dilution, a zone of increased concentration is formed). We assume that the density of the solution depends on the impurity concentration, and the kinematic viscosity ν and the diffusivity D satisfy the condition $D \ll \nu$. For example, for growth of thin films from a solution–melt of semiconducting materials, ν is of the order of 10^{-2}–10^{-3} cm^2/sec and D is of the order of 10^{-5} cm^2/sec. As a result, the fluid motion is concentrated in a thin layer adjacent to the support, and the impurity concentration differs from the average value in an even thinner layer. At high Schmidt numbers $\mathrm{Sc} = \nu/D$, a dynamic-diffusion boundary layer with thickness of order $(\mathrm{Re}^2\mathrm{Sc})^{-1/4}$ (Re is the Reynolds number) can be distinguished in the flow region. Kuznetsov and Frolovskaya [1] obtained equations of special dynamic-diffusion layers in free convection near a vertical wall for the cases where the motion is described by the classical Oberbeck–Boussinesq model and the microconvection model. The microconvection model was developed by Pukhnachev [2] to study convection in regions of small extent and in low gravitational fields or rapidly changing temperature fields. In these boundary layers, the buoyancy and viscous forces are significant and the inertial forces and the longitudinal pressure gradient are negligible. In this case, no restrictions are imposed on the Reynolds number.

Outside the dynamic-diffusion boundary layer, the impurity concentration differs only slightly from the average, and the flow pattern depends on the Reynolds number. If $\mathrm{Re} \ll \sqrt{\mathrm{Sc}}$, outside the diffusion layer, the Stokes approximation can be used. For $\mathrm{Re} \sim \sqrt{\mathrm{Sc}}$, the flow is described by the Navier–Stokes equations. If $\mathrm{Re} \gg \sqrt{\mathrm{Sc}}$, another purely dynamic boundary layer with thickness of order $(\mathrm{Sc}/\mathrm{Re}^2)^{1/4}$ is formed in the flow region. This layer is adjacent to the dynamic-diffusion layer at the inner edge and to the state of rest at the outer edge.

Thus, the flow pattern is as follows. An ultrathin dynamic-diffusion boundary layer is adjacent to the support, and the concentration in this layer is not equal to the average. At some distance from the wall there is a fluid layer in equilibrium. Between these layer there is a thin dynamic layer, in which the impurity concentration differs slightly from the average and, in addition, fluid motion is present. To determine the velocity, concentration, and pressure fields, one need first solve the problem for the dynamic-diffusion layer (without the equation for pressure) and to calculate the external representation for the velocity. After that, using this representation as the

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boundary condition, it is necessary to solve the problem for the external asymptotic form (for the particular case), to determine the dynamic-layer pressure, and, finally, to calculate the pressure in the dynamic-diffusion layer.

Kuznetsov and Frolovskaya [1] derived boundary-layer equations for the Oberbeck–Boussinesq model and the microconvection model, constructed self-similar solutions, considered the initial asymptotic form of dynamic-diffusion boundary layers, and obtained formulas for mass transfer. Frolovskaya [3] studied unsteady boundary layers, derived self-similar solutions for these layers, and showed that a countercurrent flow zone occurs in a purely dynamic boundary layer. Unlike in the case of a classical boundary layer [4], the external representation for velocity in a dynamic-diffusion layer is determined in the course of solution of the problem and not from the condition of joining with the external solution. The dynamic-layer problem differs from the classical problem in that the longitudinal velocity is specified at the inner rather than at the outer edge. All the equations obtained are nonstandard. Therefore, the question of well-posedness of the main boundary-value problems arises.

In the present paper, we derive self-similar solutions for an unsteady dynamic-diffusion boundary layer, consider initial asymptotic relations in the problem of fluid motion in a dynamic boundary layer whose inner edge is adjacent to the dynamic-diffusion layer, and study the resolvability of the boundary-value problem for a steady dynamic-diffusion boundary layer.

Self-Similar Solutions for an Unsteady Boundary Layer. The problem for an unsteady dynamic-diffusion boundary layer in the Oberbeck–Boussinesq model is to find the u and v components of the velocity $\boldsymbol{v}$, the concentration c, and the deviation from the hydrostatic pressure p in the region $y > 0$ that satisfy the initial boundary-value problem [3]:

$$\begin{gathered}
\nu\,\frac{\partial^2 u}{\partial y^2} = g\beta(c - c_\infty), \qquad \frac{1}{\rho_0}\,\frac{\partial p}{\partial y} = \nu\,\frac{\partial^2 v}{\partial y^2},\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial c}{\partial t} + u\,\frac{\partial c}{\partial x} + v\,\frac{\partial c}{\partial y} = D\,\frac{\partial^2 c}{\partial y^2},\\
c|_{t=0} = c_\infty, \qquad c|_{y=0} = f(t,x), \qquad u|_{y=0} = v|_{y=0} = 0,\\
c \underset{y\to\infty}{\longrightarrow} c_\infty, \qquad u \underset{y\to\infty}{\longrightarrow} u_\infty(t,x).
\end{gathered} \tag{1}$$

Here $\beta = \text{const} > 0$, ρ_0 and c_∞ are the average density and concentration of the solution, respectively, $f(t,x)$ is a specified function, and $u_\infty(t,x)$ is determined in the course of solution of the problem. It is seen that the velocity components and the concentration are determined independently of the pressure.

For the problem (1), self-similar solutions can be derived if $f(t,x) = c_\infty - \alpha t^{-2}$ ($\alpha = \text{const} > 0$). In this case, we seek a solution in the form

$$u = \alpha\,\frac{g\beta D}{\nu t}\,U(\xi), \quad v = 0, \quad c - c_\infty = \alpha t^{-2}(C(\xi) - 1), \quad \xi = \frac{y}{\sqrt{Dt}}.$$

To determine $U(\xi)$ and $C(\xi)$, we have the problem

$$\begin{gathered}
U'' = C - 1, \qquad C'' + (\xi/2)C' + 2(C-1) = 0,\\
U(0) = 0, \qquad C(0) = 0, \qquad U' \underset{\xi\to\infty}{\longrightarrow} 0, \qquad C \underset{\xi\to\infty}{\longrightarrow} 1,
\end{gathered}$$

whose solution is written as

$$C(\xi) = 1 + \xi(\xi^2 - 6)\mathrm{e}^{-\xi^2/4}\biggl(\gamma + 6\int\limits_\xi^\infty \frac{\mathrm{e}^{\eta^2/4}}{\eta^2(\eta^2-6)^2}\,d\eta\biggr),$$

where γ is an arbitrary constant;

$$U(\xi) = \int\limits_0^\xi \int\limits_\eta^\infty (1 - C(\omega))\,d\omega\,d\eta.$$

From this, for the external representation of the velocity, we have

$$u_\infty(t) = \alpha\,\frac{g\beta D}{\nu t}\,U_\infty, \qquad U_\infty = \int\limits_0^\infty \int\limits_\eta^\infty (1 - C(\omega))\,d\omega\,d\eta.$$

Fluid Motion for Small Times. We consider the initial process of film growth. Kuznetsov and Frolovskaya [1] studied the asymptotic form of the problem in the time interval $[0,\tau]$ as $\tau \to 0$ for the Oberbeck–Boussinesq and the microconvection model. In this case, the velocity components u and v and the concentration c

were sought in the form $u = u(t, y)$, $v \equiv 0$, and $c = c(t, y)$ under the assumption that at small times, these parameters are independent of the longitudinal coordinate (the dependence on x can develop only with time). In this case, the solution is written in quadratures.

In the Oberbeck–Boussinesq model for high Reynolds numbers, a purely dynamic boundary layer is formed in the flow region. The inner edge of this layer is adjacent to the dynamic-diffusion layer, and its outer edge neighbors the state of rest. At small times, the motion in a dynamic layer of thickness $\sqrt{\nu t}$ is described by the equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The initial boundary conditions are specified by

$$u\Big|_{t=0} = 0, \qquad u\Big|_{y=0} = u_\infty(t, x), \qquad u \underset{y\to\infty}{\longrightarrow} 0. \tag{3}$$

The function $u_\infty(t, x)$ is the external representation of the velocity, which is determined from the solution of the problem for a dynamic-diffusion boundary layer [1]:

$$u_\infty(t) = \frac{4Dg\beta(c_\infty - c_*)}{\nu}\, t\hat{U}_\infty, \qquad \hat{U}_\infty = -\int_0^\infty \int_\omega^\infty \hat{c}(\alpha)\, d\alpha\, d\omega,$$

where D, g, β, c_∞, and c_* are positive constants ($c_* < c_\infty$) and $\hat{c} = (c - c_\infty)/(c_\infty - c_*)$.

In the dynamic-layer problem, we assume that the solution is independent of the longitudinal component x. In this case, the problem reduces to the problem of boundary-layer development in time with gradual constant acceleration. We seek a solution of the problem (2), (3) in the form

$$u = \frac{4Dg\beta(c_\infty - c_*)\hat{U}_\infty}{\nu}\, tU(\zeta), \qquad v \equiv 0, \qquad \zeta = \frac{y}{2\sqrt{\nu t}}.$$

Then, to determine $U(\zeta)$, we have the problem

$$U'' + 2\zeta U' - 4U = 0,$$

$$U(0) = 1, \qquad \lim_{\zeta\to\infty} U(\zeta) = 0.$$

The solution of this problem is the function

$$U(\zeta) = (2\zeta^2 + 1)\left(1 - \frac{2}{\sqrt{\pi}} \int_0^\zeta \mathrm{e}^{-\alpha^2}\, d\alpha\right) - \frac{2}{\sqrt{\pi}}\, \zeta \mathrm{e}^{-\zeta^2}.$$

Resolvability of the Boundary-Value Problem. We study the resolvability of the boundary-value problem for the equations of a steady-state dynamic-diffusion boundary layer.

Oleinik et al. [5] studied this problem most fully using mathematical theory for boundary layers in an incompressible fluid. Problems for a temperature boundary layer were studied in [6, 7]. Dzhuraev [6] considered the boundary-value problem of continuation of a steady-state boundary layer of an incompressible fluid in forced convective flow. Khusnutdinova [7] proved the existence of a solution of the temperature boundary-layer equations for an incompressible fluid taking into account the lifting force.

In the Oberbeck–Boussinesq model, the boundary-layer equations have the form [1]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \nu \frac{\partial^2 u}{\partial y^2} = g\beta(c - c_\infty), \qquad u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}. \tag{4}$$

The main boundary-value problem is to find solutions of system (4) in the region $G = \{0 < x < A,\ 0 < y < \infty\}$ subject to the conditions

$$\begin{gathered} c\Big|_{x=0} = c_0(y), \qquad c\Big|_{y=0} = c_*, \qquad u\Big|_{y=0} = v\Big|_{y=0} = 0, \\ u \underset{y\to\infty}{\longrightarrow} u_\infty(x), \qquad c \underset{y\to\infty}{\longrightarrow} c_\infty. \end{gathered} \tag{5}$$

In the Mises variables

$$x = x, \quad \psi = \psi(x, y), \quad u = \psi_y, \quad v = -\psi_x, \quad \psi(x, 0) = 0$$

(ψ is a stream function), the main boundary-value problem for the equations of dynamic-diffusion boundary layer (4), (5) is as follows: in the region $R = \{0 < x < A,\ 0 < \psi < \infty\}$, it is required to find a solution of the system of differential equations

$$\sqrt{\omega}\,\omega_{\psi\psi} = \chi(c - c_\infty), \qquad c_x = D(\sqrt{\omega}\,c_\psi)_\psi, \tag{6}$$

subject to the boundary conditions

$$\begin{gathered} c\Big|_{x=0} = c_0(\psi), \qquad \omega\Big|_{\psi=0} = 0, \qquad c\Big|_{\psi=0} = c_*, \\ \omega(x,\psi) \underset{\psi\to\infty}{\longrightarrow} \omega_\infty(x), \qquad c(x,\psi) \underset{\psi\to\infty}{\longrightarrow} c_\infty. \end{gathered} \tag{7}$$

Here $\omega(x,\psi) = u^2(x,y)$ and $c(x,\psi) = c(x,y)$. The positive quantities χ, D, c_*, and c_∞ are assumed to be constant ($c_* < c_\infty$).

In contrast to the system considered in [7], system (6) is nonstandard. In this system, one of the equations is a quasilinear parabolic equation which degenerates at the boundary of the region, and the other equation is a second-order ordinary differential equation. In this case, one of the independent variables is treated as a parameter. In addition, the outer flow velocity $\omega_\infty(x)$ is not specified but is determined in the course of solution of the problem. Therefore, it can happen that the external representation of the solution is unbounded, which is physically unrealistic. Systems of this type have not been studied yet.

Let the initial concentration distribution $c_0(\psi)$ satisfy the following conditions:

— $c_0(\psi) > c_*$ for $\psi > 0$ and $c_0(0) = c_*$; $c_0(\psi) \to c_\infty$ for $\psi \to \infty$ and $c_0'(\psi) > 0$,

$$N/(1+\psi)^{2+\alpha} \leqslant c_\infty - c_0(\psi) \leqslant K/(1+\psi)^{2+\alpha}$$

($0 < \alpha < 1$ and N and K are positive constants);

— $c_0(\psi)$, $c_0'(\psi)$, and $c_0''(\psi)$ are bounded for $0 \leqslant \psi < \infty$ and satisfy the Hölder condition and the matching condition

$$\frac{\partial}{\partial\psi}\,(\sqrt{\omega}\,c_0'(\psi)) = 0 \quad \text{for} \quad x = 0,\ \psi = 0.$$

We introduce the following notation: $\bar{R}$ is the closure of the region R, $L(h) \equiv D(\sqrt{\omega}\,h_\psi)_\psi - h_x$, and $\Gamma = (\partial R \cap \{x = 0\}) \cup (\partial R \cap \{\psi = 0\})$.

Theorem 1. *For any finite $A > 0$ in the region R, the boundary-value problem* (6), (7) *has a solution ω, c that ehxibits the following properties: $c(x,\psi)$, $\omega(x,\psi)$, and $\omega_\psi(x,\psi)$ are continuous and bounded in $\bar{R}$; $\omega(x,\psi) > 0$ for $\psi > 0$; $k_1\psi \leqslant \omega(x,\psi) \leqslant k_2\psi$ for $0 \leqslant \psi \leqslant \psi_0$; $0 < \omega_\psi < k_3$, $-k_4 < \sqrt{\omega}\,\omega_{\psi\psi} < 0$, $0 < c(x,\psi) < k_5$ in R; $\omega(x,\psi)$ is bounded as $\psi \to \infty$; c_x, c_ψ, $c_{\psi\psi}$, and $\omega_{\psi\psi}$ are bounded in any inside closed subregion of the region R, in which the constants k_i $(i = 1,\ldots,5)$ depend on the data of the problem; ψ_0 is a small constant.*

Equation (6) for c in the region R is a parabolic equation that which degenerates at $\psi = 0$. To prove the existence of a solution of the problem (6), (7), we consider the following iterative process. Let the initial function $\omega^{(0)}(x,\psi) = \varepsilon + \tanh\,\psi$ $(\varepsilon > 0)$ be known. Knowing $\omega^{(0)}(x,\psi)$, we determine $c^{(0)}(x,\psi)$ that satisfies the equation $c_x^{(0)} = D(\sqrt{\omega^{(0)}}\,c_\psi^{(0)})_\psi$ and the boundary conditions $c^{(0)}(0,\psi) = c_0(\psi)$ and $c^{(0)}(x,0) = c_*$. Then, we move to the next iteration step. We find $\omega^{(n)}(x,\psi)$ by the formula

$$\omega^{(n)}(x,\psi) = \frac{1}{n} - \chi \int_0^\psi \int_z^\infty \frac{c^{(n-1)}(x,t) - c_\infty}{\sqrt{\omega^{(n-1)}(x,t)}}\,dt\,dz \tag{8}$$

and determine $c^{(n)}(x,\psi)$ as a solution of the equation

$$c_x^{(n)} = D(\sqrt{\omega^{(n)}}\,c_\psi^{(n)})_\psi \tag{9}$$

subject to the conditions

$$c^{(n)}\Big|_{x=0} = c_0(\psi), \qquad c^{(n)}\Big|_{\psi=0} = c_*, \tag{10}$$

where $n = 1, 2, \ldots$.

We show that the solutions $c^{(n)}(x,\psi)$ and $\omega^{(n)}(x,\psi)$ of the problem (8)–(10) exist in the region R and as $n \to \infty$, the functions $c^{(n)}(x,\psi)$ and $\omega^{(n)}(x,\psi)$ reduce to the solution of the problem (6), (7). Let M_i denote positive constants that do not depend on n.

The existence of a solution of problem (8)–(10) with the properties described in the theorem is proved by induction over n. We assume that for $n < m$, the solution $c^{(m-1)}(x,\psi)$, $\omega^{(m-1)}(x,\psi)$ of the problem (8)–(10) exists and the following inequalities are satisfied:

$$\omega^{(m-1)}(x,\psi) > 0 \text{ for } \psi \geqslant 0, \qquad \omega^{(m-1)}_{\psi\psi}(x,\psi) < 0,$$

$$1/(m-1) + M_1\psi \leqslant \omega^{(m-1)}(x,\psi) \leqslant 1/(m-1) + M_2\psi \quad (0 < \psi \leqslant \psi_0),$$

$$M_3 < \omega^{(m-1)}(x,\psi) < M_4 \quad (\psi \geqslant \psi_0), \qquad 0 < \omega^{(m-1)}_{\psi}(x,\psi) < M_5, \tag{11}$$

$$c_* + \varphi(\psi)\mathrm{e}^{-\beta x} \leqslant c^{(m-1)}(x,\psi) \leqslant c_\infty,$$

$$-\frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{(1+\psi)^{2+\alpha}} \leqslant c^{(m-1)}(x,\psi) - c_\infty \leqslant -\frac{M_7(1+x)^{-\gamma}}{(1+\psi)^{2+\alpha}},$$

$$c^{(m-1)}(x,\psi) \leqslant c_* + M_8\sqrt{\psi} \quad (0 < \psi < \psi_0)$$

[$\varphi(\psi)$, α, β, λ, and γ are determined below]. We show that these estimates are valid for $n = m$ as well.

With allowance for estimates (11), from (8), we find that $\omega^{(m)} \geqslant 1/m > 0$ as $\psi \geqslant 0$.

Lemma 1. *For the function $\omega^{(m)}(x,\psi)$, the following inequality is valid:*

$$1/m + M_1\psi \leqslant \omega^{(m)}(x,\psi) \leqslant 1/m + M_2\psi. \tag{12}$$

Here $0 \leqslant \psi \leqslant \psi_0$.

Proof. From (8), using (11), we obtain

$$\omega^{(m)}(x,\psi) \geqslant \frac{1}{m} + \chi \int_0^\psi \int_z^\infty \frac{M_7(1+x)^{-\gamma}}{\sqrt{\omega^{(m-1)}(x,t)}(1+t)^{2+\alpha}}\,dt\,dz$$

$$\geqslant \frac{1}{m} + \frac{\chi M_7(1+x)^{-\gamma}}{\sqrt{M_4}} \int_0^\psi \int_z^\infty \frac{1}{(1+t)^{2+\alpha}}\,dt\,dz \geqslant \frac{1}{m} + \frac{\chi M_7(1+A)^{-\gamma}}{\sqrt{M_4}\alpha(1+\alpha)}\Big(1 - \frac{1}{(1+\psi)^\alpha}\Big) \geqslant \frac{1}{m} + M_1\psi$$

for small ψ, and

$$\omega^{(m)}(x,\psi) \leqslant \frac{1}{m} + \chi \int_0^\psi \int_z^\infty \frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{\sqrt{\omega^{(m-1)}(x,t)}(1+t)^{2+\alpha}}\,dt\,dz$$

$$\leqslant \frac{1}{m} + \frac{\chi M_6\mathrm{e}^{1/(1-\lambda x)}}{\sqrt{M_3}} \int_0^\psi \int_z^\infty \frac{1}{(1+t)^{2+\alpha}}\,dt\,dz \leqslant \frac{1}{m} + \frac{\chi M_6\mathrm{e}}{\sqrt{M_3}\alpha(1+\alpha)}\Big(1 - \frac{1}{(1+\psi)^\alpha}\Big) \leqslant \frac{1}{m} + M_2\psi$$

for $0 \leqslant \psi \leqslant \psi_0$. Lemma 1 is proved.

From (12) it follows that $\omega^{(m)} > M_3$ for $\psi \geqslant \psi_0$. From (8), using this estimate and (11), we have $\omega^{(m)} < M_4$. Indeed,

$$\omega^{(m)}(x,\psi) \leqslant \frac{1}{m} + \chi \int_0^\psi \int_z^\infty \frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{\sqrt{\omega^{(m-1)}(x,t)}(1+t)^{2+\alpha}}\,dt\,dz$$

$$\leqslant \frac{1}{m} + \frac{\chi M_6\mathrm{e}}{\sqrt{M_3}} \int_0^\psi \int_z^\infty \frac{1}{(1+t)^{2+\alpha}}\,dt\,dz = \frac{1}{m} + \frac{\chi M_6\mathrm{e}}{\alpha(1+\alpha)\sqrt{M_3}}\Big(1 - \frac{1}{(1+\psi)^\alpha}\Big)$$

$$\leqslant 1 + \frac{\chi M_6\mathrm{e}}{\alpha(1+\alpha)\sqrt{M_3}}\Big(1 - \frac{1}{(1+\psi)^\alpha}\Big) < M_4.$$

Thus, $M_3 < \omega^{(m)}(x,\psi) < M_4$ for $\psi \geqslant \psi_0$, i.e., the function $\omega^{(m)}(x,\psi)$ takes finite values as $\psi \to \infty$.

From the estimates (11), it also follows that $\omega_\psi^{(m)} > 0$ and $\omega_{\psi\psi}^{(m)} < 0$. We show that $\omega_\psi^{(m)}(x,\psi)$ is bounded from above. From (8), taking into account the estimates (11), we obtain

$$\omega_\psi^{(m)}(x,\psi) = -\chi \int\limits_\psi^\infty \frac{c^{(m-1)}(x,t) - c_\infty}{\sqrt{\omega^{(m-1)}(x,t)}}\, dt$$

$$\leqslant \chi \int\limits_\psi^\infty \frac{M_6 \mathrm{e}^{1/(1-\lambda x)}}{\sqrt{\omega^{(m-1)}(x,t)}(1+t)^{2+\alpha}}\, dt \leqslant \frac{\chi M_6 \mathrm{e}}{(1+\alpha)\sqrt{M_3}} \frac{1}{(1+\psi)^{1+\alpha}} < M_5.$$

Lemma 2. *If a positive solution $c^{(m)}(x,\psi)$ of the problem* (9), (10) *exists in the region R, the following a priory estimate holds:*

$$c^{(m)}(x,\psi) \geqslant c_* + \varphi(\psi)\mathrm{e}^{-\beta x}. \tag{13}$$

Here

$$\varphi(\psi) = \begin{cases} A_1\psi & \text{for} \quad \psi \in (0,1), \\ A_1(1+\tanh\,(\psi-1)) & \text{for} \quad \psi \in (1,\infty) \end{cases}$$

$[A_1 = \min\{c_0(1)/3, c_0'(0)/3\}]$ *and* $\beta = \mathrm{const} > 0$.

Proof. We consider the function $Q^{(m)} = c^{(m)}(x,\psi) - c_* - \varphi(\psi)\mathrm{e}^{-\beta x}$. From this, expressing $c^{(m)}(x,\psi)$ and substituting it into (9), we obtain the equation for $Q^{(m)}$:

$$L(Q^{(m)}) = -\mathrm{e}^{-\beta x}\Big[\beta\varphi(\psi) + D\sqrt{\omega^{(m)}}\,\varphi''(\psi) + D\omega_\psi^{(m)}\varphi'(\psi)/(2\sqrt{\omega^{(m)}})\Big]. \tag{14}$$

For $0 < \psi \leqslant \psi_0 < 1$, we have

$$L(Q^{(m)}) = -\mathrm{e}^{-\beta x} A_1\Big[\beta\psi + D\omega_\psi^{(m)}/(2\sqrt{\omega^{(m)}})\Big].$$

For $\psi > 1$, the function $\varphi(\psi) > A_1 > 0$, $0 < \varphi'(\psi) < A_1$, and $-A_1 < \varphi''(\psi) \leqslant 0$. If β is large enough, the expression in square brackets in (14) is positive. Hence, $L(Q^{(m)}) \leqslant 0$. On the boundary $\Gamma = \{x = 0,\ \psi = 0\}$, we have

$$Q^{(m)}\Big|_{x=0} = c_0(\psi) - c_* - \varphi(\psi) \geqslant 0, \qquad Q^{(m)}\Big|_{\psi=0} = 0.$$

If $c^{(m)}(x,\psi)$ exists in the region R, $Q^{(m)}|_\Gamma \geqslant 0$, $|Q^{(m)}| < M_0$, and $L(Q^{(m)}) \leqslant 0$. In the region R, the function $Q^{(m)}(x,\psi)$, which is nonnegative on Γ, satisfies the linear parabolic equation $L(Q^{(m)}) = F$, where $F \leqslant 0$. By the maximum principle [5, 8], $Q^{(m)} \geqslant 0$ everywhere in R, and, hence, estimate (13) holds. Lemma 2 is proved.

By virtue of the estimate $\omega^{(m)}(x,0) > 0$, Eq. (9) for $c^{(m)}(x,\psi)$ is a linear parabolic equation, and the well-known theorems of the existence of a solution of the first boundary-value problem [8] can be applied to this equation. This solution has first derivatives with respect to ψ and x and the second derivative with respect to ψ, which satisfy the Hölder condition for the closed region $\bar{R}$. We obtain estimates for $c^{(m)}(x,\psi)$ in the region R that are uniform with respect to m.

Lemma 3. *In the region R, the following estimate holds:*

$$c^{(m)}(x,\psi) \leqslant c_\infty.$$

Proof. In Eq. (9), we substitute $c^{(m)}(x,\psi) = c_\infty + \bar{c}^{(m)}\mathrm{e}^{\alpha x}$, where $\alpha = \mathrm{const} > 0$. Then, $\bar{c}^{(m)}$ satisfies the equation $L(\bar{c}^{(m)}) = \alpha\bar{c}^{(m)}$. At the positive maximum of the function $\bar{c}^{(m)} = (c^{(m)} - c_\infty)\mathrm{e}^{-\alpha x}$, if it is reached within the region R, the inequality $\bar{c}^{(m)} \leqslant 0$ is valid. On the boundary Γ, we have

$$\bar{c}^{(m)}\Big|_{x=0} = c_0(\psi) - c_\infty \leqslant 0, \qquad \bar{c}^{(m)}\Big|_{\psi=0} \leqslant 0.$$

Hence, $\bar{c}^{(m)}(x,\psi) \leqslant 0$ everywhere in R; therefore, $c^{(m)}(x,\psi) \leqslant c_\infty$ everywhere in R. The lemma is proved.

Lemma 4. *In the region R, for $\psi \geqslant \psi_0$, the following estimates hold:*

$$-\frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{(1+\psi)^{2+\alpha}} \leqslant c^{(m)}(x,\psi) - c_\infty \leqslant -\frac{M_7(1+x)^{-\gamma}}{(1+\psi)^{2+\alpha}}, \tag{15}$$

where $0 < \alpha < 1$, $\lambda = \mathrm{const} > 0 \quad (\lambda \neq 1/x)$, and $\gamma = \mathrm{const} > 0$.

Proof. Let $S^{(m)}(x,\psi)=c^{(m)}(x,\psi)-c_\infty$. Then, $S^{(m)}(x,\psi)$ satisfies the equation $L(S^{(m)})=0$. We consider the function

$$Q^{(m)}(x,\psi)=S^{(m)}(x,\psi)+M_6\mathrm{e}^{1/(1-\lambda x)}/(1+\psi)^{2+\alpha}$$

and

$$L(Q^{(m)})=\Big[\frac{D(2+\alpha)(3+\alpha)\sqrt{\omega^{(m)}}}{(1+\psi)^2}-\frac{D(2+\alpha)\omega_\psi^{(m)}}{2(1+\psi)\sqrt{\omega^{(m)}}}-\frac{\lambda}{(1-\lambda x)^2}\Big]\frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{(1+\psi)^{2+\alpha}}.$$

If λ is large enough, $L(Q^{(m)})\leqslant 0$. On the boundary $\Gamma_1=\{x=0,\ \psi=\psi_0\}$, we have

$$Q^{(m)}\Big|_{x=0}=c_0(\psi)-c_\infty+\frac{M_6\mathrm{e}}{(1+\psi)^{2+\alpha}}\geqslant 0,\quad Q^{(m)}\Big|_{\psi=\psi_0}=c(x,\psi_0)-c_\infty+\frac{M_6\mathrm{e}^{1/(1-\lambda x)}}{(1+\psi_0)^{2+\alpha}}\geqslant 0$$

for a large M_6. Then, by virtue of the inequalities $Q^{(m)}|_{\Gamma_1}\geqslant 0$ and $|Q^{(m)}|\leqslant M_9, L(Q^{(m)})\leqslant 0$, $(x,\psi)\in R$, the function $Q^{(m)}(x,\psi)$ is nonpositive over the entire region R.

Let us prove the upper estimate. We consider the function

$$P^{(m)}(x,\psi)=S^{(m)}(x,\psi)+M_7(1+x)^{-\gamma}/(1+\psi)^{2+\alpha},$$

$$L(P^{(m)})=\Big[\frac{D(2+\alpha)(3+\alpha)\sqrt{\omega^{(m)}}}{(1+\psi)^2}-\frac{D(2+\alpha)\omega_\psi^{(m)}}{2(1+\psi)\sqrt{\omega^{(m)}}}+\frac{\gamma}{1+x}\Big]\frac{M_7(1+x)^{-\gamma}}{(1+\psi)^{2+\alpha}}.$$

For sufficiently large γ, $L(P^{(m)})\geqslant 0$. If $1/\gamma$ and M_7 are sufficiently small, then $P^{(m)}\leqslant 0$ at $x=0$ and $\psi=\psi_0$. Indeed,

$$P^{(m)}\Big|_{x=0}=c_0(\psi)-c_\infty+\frac{M_7}{(1+\psi)^{2+\alpha}}\leqslant 0,\quad P^{(m)}\Big|_{\psi=\psi_0}=c(x,\psi_0)-c_\infty+\frac{M_7(1+x)^{-\gamma}}{(1+\psi_0)^{2+\alpha}}\leqslant 0.$$

Since $P^{(m)}\Big|_{\Gamma_1}\leqslant 0$ and $|P^{(m)}|\leqslant M_{10}$, $L(P^{(m)})\geqslant 0$, then, by the maximum principle, $P^{(m)}(x,\psi)\leqslant 0$ over the entire R, and, hence, estimates (15) are correct. The lemma is proved.

Lemma 5. *In the region $R_1=\{0<x<A, 0<\psi<\psi_0)$, the following estimate is valid:*

$$c^{(m)}(x,\psi)\leqslant c_*+M_8\sqrt{\psi}. \tag{16}$$

Proof. We consider the function $S^{(m)}(x,\psi)=c^{(m)}(x,\psi)-c_*-M_8\sqrt{\psi}$ satisfying the equation

$$L(S^{(m)})=\frac{D}{4\sqrt{\omega^{(m)}}}\Big[\frac{\omega^{(m)}}{\psi^2}-\frac{\omega_\psi^{(m)}}{\psi}\Big]M_8\sqrt{\psi}.$$

We show that the expression in the square brackets is nonnegative. Taylor expansion of the functions $\omega^{(m)}(x,\psi)$ and $\omega_\psi^{(m)}(x,\psi)$ in the neighborhood of zero yields

$$\omega^{(m)}(x,\psi)=\omega^{(m)}(x,0)+\omega_\psi^{(m)}(x,0)\psi+\omega_{\psi\psi}^{(m)}(x,0)\psi^2/2+o(\psi^2),$$
$$\omega_\psi^{(m)}(x,\psi)=\omega_\psi^{(m)}(x,0)+\omega_{\psi\psi}^{(m)}(x,0)\psi+o(\psi).$$

Then,

$$\omega^{(m)}(x,\psi)-\psi\omega_\psi^{(m)}(x,\psi)=1/m-\omega_{\psi\psi}^{(m)}(x,0)\psi^2/2+o(\psi^2)\geqslant 0$$

because $\omega_{\psi\psi}^{(m)}<0$. Therefore, for small ψ, $L(S^{(m)})\geqslant 0$. On the boundary, we have

$$S^{(m)}\Big|_{x=0}=c_0(\psi)-c_*-M_8\sqrt{\psi}\leqslant 0,$$

$$S^{(m)}\Big|_{\psi=0}=0,\qquad S^{(m)}\Big|_{\psi=\psi_0}=c^{(m)}(x,\psi_0)-c_*-M_8\sqrt{\psi_0}\leqslant 0$$

if M_8 is sufficiently large.

Since $S^{(m)}|_{\partial R_1}\leqslant 0$, $|S^{(m)}|\leqslant M_{11}$, and $L(S^{(m)})\geqslant 0$, then, by the maximum principle, $S^{(m)}(x,\psi)\leqslant 0$ in R_1, i.e., the estimate (16) is correct. Lemma 5 is proved.

The iterative process can continue for an indefinitely long time. In every iteration step, there is a solution of problem (8)–(10) with the required properties.

Let $\delta > 1$ be an arbitrary number. We consider the region $R_\delta = \{0 < x < A,\ 1/\delta < \psi < \delta\}$. In this region, $K_1 < \sqrt{\omega^{(n)}} < K_2$, where K_i are positive constants that depend on δ but do not depend on n, and $\omega_\psi^{(n)}$ is uniformly bounded in n. Because the coefficients of Eq. (9) are separated uniformly (in n) from zero and infinity, the sequence $\{c^{(n)}(x,\psi)\}$ is compact. Therefore, we can distinguish a subsequence $\{c^{(n_k)}(x,\psi)\}$ that converges uniformly to the function $c(x,\psi)$ which is continuous in R_δ, as $n_k \to \infty$. The derivatives $\partial c^{(n)}/\partial x$, $\partial c^{(n)}/\partial \psi$, and $\partial^2 c^{(n)}/\partial \psi^2$ converge uniformly to the corresponding derivatives of the function $c(x,\psi)$ in each finite region $[0,A] \times [1/\delta, \delta]$. By virtue of the arbitrariness of δ, the limiting function $c(x,\psi)$ satisfies the second equation in (6) over the entire region R.

Finally, we prove that $c(x,\psi)$ satisfies conditions (7). According to the estimates (13) and (16), we have

$$c_* \leqslant c(x,\psi) \leqslant c_* + M_8\sqrt{\psi}. \tag{17}$$

In inequality (17), passage to the limit as $\psi \to 0$ proves that $\lim\limits_{\psi\to 0} c(x,\psi) = c_*$ exists. As in [6], it can be shown that the function $c(x,\psi)$ also satisfies the last condition in (7).

From the estimates $M_1\psi \leqslant \omega(x,\psi) \leqslant M_2\psi$, it follows that $\lim\limits_{\psi\to 0} \omega(x,\psi) = 0$.

Thus, the functions $\omega(x,\psi)$ and $c(x,\psi)$ are a solution of the boundary-value problem (6), (7). From estimates (17), it follows that the function $c(x,\psi)$ is continuous up to the boundary. The continuity and boundedness of the derivatives c_x, c_ψ, and $c_{\psi\psi}$ in every inside subregion of the region R follow from the properties of solutions of linear parabolic equations. The continuity of ω and ω_ψ up to the boundary and the boundedness of $\omega_{\psi\psi}$ in every inside closed subregion of the region R follows from the definition of the function $\omega(x,\psi)$ (8) and the properties of the function $c(x,\psi)$. The theorem is proved. After converting to the physical variables (x,y), the solution in the Mises variables will have all the necessary properties [5].

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